

Quantum Quench in the Transverse Field Ising Chain II: Stationary State Properties

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Abstract. We consider the stationary state properties of the reduced density matrix as well as spin-spin correlation functions after a sudden quantum quench of the magnetic field in the transverse field Ising chain. We demonstrate that stationary state properties are described by a generalized Gibbs ensemble. We discuss the approach to the stationary state at late times.

1. Introduction

This is the second of two papers on the quench dynamics of the transverse field Ising chain. In the first part of our work [1], which in the following we will refer to as “paper I”, we focussed on the time dependence of the longitudinal spin correlations. The present manuscript gives a detailed account of properties in the *stationary state* at infinite times after the quench. An important motivation for studying problems of nonequilibrium time evolution in isolated quantum systems is provided by recent experiments on trapped ultra-cold atomic gases [2, 3, 4, 5, 6, 7]. These experiments suggest that observables such as multi-point correlation functions generically relax to time independent values. Such a behaviour at first appears quite surprising, because unitary time evolution maintains the system in a pure state at all times. The resolution of this apparent paradox is that in the thermodynamic limit, (finite) *subsystems* can and do display correlations characteristic of a mixed state, namely the one obtained by tracing out the degrees of freedom outside the subsystem itself. In physical terms this means that the system acts as its own bath. An important question is how to characterize the reduced density matrix describing the stationary behaviour. Intuitively one may expect that conservation laws will play an important role, which in turn poses the question whether quantum integrable systems exhibit qualitatively different stationary behaviours when compared to generic, nonintegrable ones. These issues have been addressed in a number of recent works [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42]. Most of these studies are compatible with the widely held belief (see e.g. [8] for a comprehensive summary) that the reduced density matrix of any finite subsystem (which determines correlation functions of all local observables within the subsystem) of an infinite system can be described in terms of either an effective thermal (Gibbs) distribution or a so-called generalized Gibbs ensemble (GGE) [9]. It has been conjectured that the latter arises for integrable models, while the former is obtained for generic systems. Evidence supporting this view has been obtained in a number of examples [9, 10, 12, 13, 14, 15, 16, 19, 21, 22, 23]. On the other hand, several numerical studies [24, 26, 27, 37, 38] suggest that the full picture may well be more complex. Moreover, open questions remain even with regard to the very existence of stationary states. For example, the order parameter of certain mean-field models have been shown to display persistent oscillations [43, 44, 45, 46, 47, 48]. Non-decaying oscillations have also been observed numerically [26] in some non-integrable one-dimensional systems. This has given rise to the concept of “weak thermalization”, which refers to a situation where only time-averaged quantities are thermal.

We will show in this manuscript that for a quench of the magnetic field in the transverse field Ising chain, the reduced density matrix is described by a GGE. This establishes that *any local observable* is described by the GGE. However we will also show that while some two-point observables approach their asymptotic values relatively quickly, others will do so only after times exponentially large in the separation between the two points. In practice this precludes experimental or numerical detection of stationary behaviour for these observables.

1.1. Quench protocol and observables

In the following we focus on a *global quantum quench* of the magnetic field in the Ising Hamiltonian

$$H(h) = -J \sum_{j=1}^L \left[\sigma_j^x \sigma_{j+1}^x + h \sigma_j^z \right], \quad (1)$$

where σ_j^α are the Pauli matrices at site j , we assume $J > 0$, $h > 0$ and we impose periodic boundary conditions $\sigma_{L+1}^\alpha = \sigma_1^\alpha$. We assume that the many-body system is prepared in the ground

state $|\Psi_0\rangle$ of Hamiltonian $H(h_0)$. At time $t = 0$ the field h_0 is changed instantaneously to a different value h and one then considers the unitary time evolution of the system characterized by the new Hamiltonian $H(h)$, i.e. the initial state $|\Psi_0\rangle$ evolves as

$$|\Psi_0(t)\rangle = e^{-itH(h)}|\Psi_0\rangle. \quad (2)$$

The above protocol corresponds to an experimental situation [2, 4, 5], in which a system parameter has been changed on a time scale that is small compared to all characteristic time scales present in the system.

In equilibrium at zero temperature (and in the thermodynamic limit) the Ising model (1) exhibits ferromagnetic ($h < 1$) and paramagnetic ($h > 1$) phases, separated by a quantum critical point at $h_c = 1$. The order parameter for the corresponding quantum phase transition is the ground state expectation value $\langle\sigma_j^x\rangle$. The Hamiltonian (1) can be diagonalized by a Jordan-Wigner transformation, which maps the model to spinless fermions with local annihilation operators c_j , followed by a Fourier transform and, finally, a Bogoliubov transformation. In terms of the momentum space Bogoliubov fermions α_k the Hamiltonian is diagonal:

$$H(h) = \sum_k \varepsilon_h(k) \alpha_k^\dagger \alpha_k, \quad \varepsilon_h(k) = 2J\sqrt{1 + h^2 - 2h\cos(k)}. \quad (3)$$

Details and precise definitions are given in Appendix A of paper I.

In the following we focus on the one and two-point functions of the order parameter

$$\rho^x(t) = \frac{\langle\Psi_0(t)|\sigma_\ell^x|\Psi_0(t)\rangle}{\langle\Psi_0(t)|\Psi_0(t)\rangle}, \quad (4)$$

$$\rho^{xx}(\ell, t) = \frac{\langle\Psi_0(t)|\sigma_{j+\ell}^x\sigma_j^x|\Psi_0(t)\rangle}{\langle\Psi_0(t)|\Psi_0(t)\rangle}, \quad \rho_c^{xx}(\ell, t) = \rho^{xx}(\ell, t) - (\rho^x(t))^2, \quad (5)$$

the transverse spin correlators

$$\rho^z(t) = \frac{\langle\Psi_0(t)|\sigma_\ell^z|\Psi_0(t)\rangle}{\langle\Psi_0(t)|\Psi_0(t)\rangle}, \quad \rho_c^{zz}(\ell, t) = \frac{\langle\Psi_0(t)|\sigma_{j+\ell}^z\sigma_j^z|\Psi_0(t)\rangle}{\langle\Psi_0(t)|\Psi_0(t)\rangle} - (\rho^z(t))^2, \quad (6)$$

and the reduced density matrix of subsystem A

$$\rho_A(t) = \text{Tr}_{\bar{A}}|\Psi_0(t)\rangle\langle\Psi_0(t)|. \quad (7)$$

Here $A \cup \bar{A}$ is the entire system and in practice we will take A to consist of ℓ consecutive sites along the chain. $\text{Tr}_{\bar{A}}$ denotes a trace in the space of states describing the lattice sites in \bar{A} .

Following paper I, we divide the time evolution of a two-point function for a fixed distance ℓ between the operator insertions into three regimes, which are determined by the propagation velocity $v(k) = \frac{d\varepsilon_h(k)}{dk}$ of elementary excitations of the post-quench Hamiltonian. For a given final magnetic field h , the maximal propagation velocity is

$$v_{\max} = \max_{k \in [-\pi, \pi]} |\varepsilon'_h(k)| = 2J \min[h, 1]. \quad (8)$$

The three different regimes are:

- Short-times $v_{\max}t \ll \ell$.
- Intermediate times $v_{\max}t \sim \ell$. This regime is of particular importance for both experiments and numerical computations. A convenient way of describing this regime is to consider evolution along a particular “ray” $\kappa\ell = v_{\max}t$ in space-time, see Fig. 1. In order to obtain an accurate description of the dynamics at a particular point along this ray, one may then construct an asymptotic expansion in the single variable ℓ around the *space-time scaling limit* $v_{\max}t, \ell \rightarrow \infty$, κ fixed.

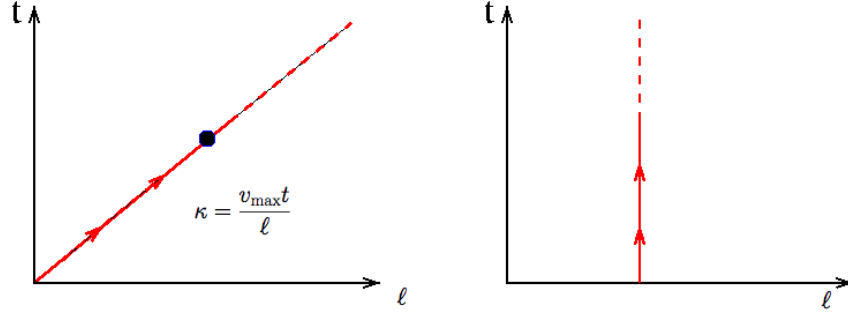


Figure 1. Left panel: for intermediate times $v_{\max}t \sim \ell$ the behaviour of $\rho^{\alpha\alpha}(\ell, t)$ is most conveniently determined by considering its asymptotic expansion around infinity (“space-time scaling limit”) along the ray $v_{\max}t = \kappa\ell$. This viewpoint is appropriate for any large, finite t and ℓ . Right panel: the asymptotic late-time regime is reached by considering time evolution at fixed ℓ . To describe this regime one should consider an asymptotic expansion of $\rho^{\alpha\alpha}(\ell, t)$ around $t = \infty$ at fixed ℓ .

- Late times $v_{\max}t \gg \ell$. This includes the limit $t \rightarrow \infty$ at fixed but large ℓ . In this regime it is no longer convenient to consider evolution along a particular ray in space-time. In order to obtain accurate results for the late time dynamics, one should construct an asymptotic expansion in t around infinity, see Fig. 1.

It is important to note that in general taking $\kappa \rightarrow \infty$ in the space-time scaling limit does not necessarily reproduce the late time behaviour at fixed, asymptotically large ℓ . In other words, in general we have

$$\lim'_{\kappa \rightarrow \infty} \lim'_{\substack{t, \ell \rightarrow \infty \\ \kappa \text{ fixed}}} \rho_c^{\alpha\alpha}(\ell, t) \neq \lim'_{\ell \rightarrow \infty} \lim'_{t \rightarrow \infty} \rho_c^{\alpha\alpha}(\ell, t), \quad (9)$$

where \lim' denotes the leading term in an asymptotic expansion around the limiting point. We will show that the limits *do commute* for $\rho^{xx}(\ell, t)$ for most but not all quenches, but they never commute for $\rho_c^{zz}(\ell, t)$.

1.2. Longitudinal versus Transverse Correlators

A global quantum quench of the transverse field in the Ising model is an essentially ideal testing ground for many ideas related to thermalization [49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61]. An example is the issue to what extent the late time behaviour after a quench may depend on the observable under consideration. In this regard, one may expect the *locality* of observables relative to the elementary excitations of the model to play a role [49, 26, 50]. In the Ising chain, the transverse spin operator σ_j^z is local with respect to the fermionic degrees of freedom c_j , while the order parameter σ_j^x is in general non-local. The non-locality makes the longitudinal correlators difficult to analyze and general analytic results have been reported only recently in our short communication [51] (the stationary properties in the special cases $h_0 = 0, \infty$ were obtained before in Ref. [55]). The results reported in [51] for the correlation length in the stationary state were found to be described by an appropriately defined GGE.

In contrast to order parameter correlators, the time evolution of one and two point functions of the transverse spins σ_m^z is straightforward to analyze (as the latter are fermion bilinears) and

has been known since 1970 [52]. In spite of its simplicity, the connected two-point function $\rho_c^{zz}(\ell, t)$ (corresponding to density correlations in gases) exhibits an interesting and quite general phenomenon, namely a cross-over in the relaxational behaviour at an exponentially large time scale $t_{\text{cross}} \sim e^\ell$. As a consequence the truly stationary regime of $\rho_c^{zz}(\ell, t)$ is observed only at very late times $t > t_{\text{cross}}$, which in general is a serious limitation.

1.3. The quench variables

As shown in Appendix A of paper I, both the initial and final Hamiltonians can be diagonalized by combined Jordan-Wigner and Bogoliubov transformations with Bogoliubov angles θ_k^0 and θ_k respectively

$$e^{i\theta_k} = \frac{h - e^{ik}}{\sqrt{1 + h^2 - 2h \cos k}}, \quad e^{i\theta_k^0} = \frac{h_0 - e^{ik}}{\sqrt{1 + h_0^2 - 2h_0 \cos k}}. \quad (10)$$

The corresponding Bogoliubov fermions are related by a linear transformation characterized by the difference $\Delta_k = \theta_k - \theta_k^0$. In order to parametrize the quench it is useful to introduce the quantity

$$\cos \Delta_k = \frac{hh_0 - (h + h_0) \cos k + 1}{\sqrt{1 + h^2 - 2h \cos(k)} \sqrt{1 + h_0^2 - 2h_0 \cos(k)}}. \quad (11)$$

We note that $\cos \Delta_k$ is invariant under the two transformations

$$(h_0, h) \rightarrow (h, h_0) \quad \text{and} \quad (h_0, h) \rightarrow \left(\frac{1}{h_0}, \frac{1}{h} \right). \quad (12)$$

However, we stress that the quantum quench itself is not invariant under the maps (12).

1.4. Generalized Gibbs ensemble

The density matrix of a GGE can be cast in the form [62]

$$\rho_{\text{GGE}} = \frac{1}{Z_{\text{GGE}}} e^{-\sum_m \lambda_m I_m}, \quad (13)$$

where I_m are some integrals of motion and Z_{GGE} ensures the normalization condition $\text{Tr} \rho_{\text{GGE}} = 1$. In Ref. [9], Rigol, Dunjko, Yurovsky and Olshanii proposed that an integrable system after a quantum quench in the infinite time limit is in fact described by a GGE, where the I_m represent a complete set of independent integrals of motion and the Lagrange multipliers λ_m are fully determined by the initial state $|\Psi_0\rangle$ through the conditions

$$\text{Tr}[\rho_{\text{GGE}} I_m] = \langle \Psi_0 | I_m | \Psi_0 \rangle. \quad (14)$$

As we are particularly interested in the thermodynamic limit, we will employ the following, somewhat narrower, definition of a GGE for the description of stationary properties after quantum quenches.

- First, as the entire system will be in a pure state at all times, it cannot be described by a density matrix corresponding to a mixed state. We therefore take ρ_{GGE} to be the *reduced density matrix* of an infinitely large subsystem A in the thermodynamic limit. From this reduced density matrix all multipoint correlation functions can be obtained. Moreover, one can extract the reduced density matrix of any finite subsystem A_1 by [15, 16]

$$\rho_{A_1}(t = \infty) = \text{Tr}_{\bar{A}_1} \rho_{\text{GGE}}. \quad (15)$$

Considering an infinitely large subsystem A has the advantage that the density matrix defining the GGE can be expressed in a simple way even for more complicated, interacting integrable models, whereas the corresponding formulation for a finite subsystem becomes difficult. However, in cases where the subsystem A_1 consists of several disjoint blocks, the reduced density matrix will not have a simple form like (16) even in the absence of interactions [63].

The precise sequence of limits we have in mind is the following. The entire system is decomposed in a subsystem A and its complement \bar{A} . We then take the thermodynamic limit, keeping A fixed. Finally we take the limit of A becoming infinite itself. The operator ρ_{GGE} is the reduced density matrix of subsystem A in this limit.

The importance of considering the reduced density matrix when applying GGE ideas has been emphasized previously in Refs. [15, 16, 19, 26, 27].

- Another crucial point, that perhaps has not yet received the attention it deserves, concerns the issue of what integrals of motion I_m ought to be included in the definition (14) of the GGE density matrix. Indeed, any quantum system, integrable or not, has many integrals of motion. For example the projectors $I_n = |\psi_n\rangle\langle\psi_n|$ on Hamiltonian eigenstates $H|\psi_n\rangle = E_n|\psi_n\rangle$ are integrals of motion $[H, I_n] = [I_m, I_n] = 0$. Clearly, such conservation laws cannot play any role in determining the late time behaviour after a quantum quench. Following [64] we therefore propose to include only *local* integrals of motion in (14). These are characterized by arising from an integral (or a sum in the case of lattice models) of a local current density $J_n(x)$ as $I_n = \int dx J_n(x)$, in the same spirit of the well known Noether theorem in quantum field theory. This locality requirement is similar in spirit to the additivity requirement of Ref. [65].

With these remarks in mind, the generalized Gibbs ensemble for the Ising chain is then defined by the density matrix [9]

$$\rho_{\text{GGE}} = \frac{1}{Z_{\text{GGE}}} \exp \left(- \sum_k \beta_k \varepsilon_k \alpha_k^\dagger \alpha_k \right), \quad (16)$$

and the Lagrange multiplier β_k are fixed by the initial state $|\Psi_0\rangle$ through the equations

$$\langle \Psi_0 | \alpha_k^\dagger \alpha_k | \Psi_0 \rangle = \text{Tr}[\rho_{\text{GGE}} \alpha_k^\dagger \alpha_k]. \quad (17)$$

For a quench of the magnetic field in the Ising chain, these are easily solved with the results

$$\beta_k \varepsilon_k = 2 \text{arctanh}(\cos \Delta_k). \quad (18)$$

However, the integrals of motion $n_k = \alpha_k^\dagger \alpha_k$ used in (16) are *non-local* in space. A priori this is a serious problem as the integrals of motion defining a GGE must be local, as we pointed out above. However, as shown in Ref. [64], the prescription (16) works, because in the particular case of the Ising model the local integrals of motion can be expressed as complicated *linear* combinations of $n_k = \alpha_k^\dagger \alpha_k$. Hence for an infinite subsystem (16) is equivalent to a GGE defined using the local integrals of motion.

1.5. Organization of the manuscript.

The present manuscript is organized as follows. In section 2 we give a detailed summary of our main results. In Sec. 3 we show that the reduced density matrix is described by a GGE. In Sec. 4 we compute the stationary values of the longitudinal two-point correlation function, while in Sec. 5 we study transverse correlations and their approach to the GGE. In Appendix A we review some

mathematical theorems we need to evaluate the asymptotic behaviour of longitudinal correlation functions.

2. Summary of Results

2.1. Reduced Density Matrix in the Stationary State

Given that we start the system out in a pure state $|\Psi_0\rangle$, its full density matrix is the one-dimensional projector

$$\rho(t) = |\Psi_0(t)\rangle\langle\Psi_0(t)|. \quad (19)$$

The reduced density matrix of a subsystem A is then obtained as

$$\rho_A(t) = \text{Tr}_{\bar{A}}(\rho(t)), \quad (20)$$

where \bar{A} is the complement of A . We have obtained the following result for $\rho_A(t = \infty)$: in the thermodynamic limit, for the case where A is itself infinite, the reduced density matrix in the stationary state is equivalent to that of the generalized Gibbs ensemble (16)

$$\rho_A(t = \infty) = \rho_{\text{GGE}}. \quad (21)$$

In particular, this implies that arbitrary *local* multi-point spin correlation functions within subsystem A can be evaluated as averages within the GGE. However, the calculation of such averages represents itself a formidable problem. We have determined these averages for the most important cases of the two-point functions $\rho^{xx}(\ell, t = \infty)$ and $\rho^{zz}(\ell, t = \infty)$ and present explicit expressions below. We stress that although σ_n^x is not local in terms of fermions, correlation functions involving σ_n^x can nevertheless be calculated from ρ_{GGE} . For correlation functions involving an even number of σ_n^x 's this is because the Jordan-Wigner strings cancel and one is left with an expression that involves only fermions within subsystem A . Correlation functions involving an odd number of σ_n^x 's vanish as a result of the \mathbb{Z}_2 symmetry, which is restored in the GGE.

2.2. Longitudinal Correlators in the Stationary State

We find that the infinite time limit of the two-point function $\lim_{t \rightarrow \infty} \rho^{xx}(\ell, t)$ for fixed but asymptotically large ℓ is given by

$$\rho^{xx}(\ell \gg 1, t = \infty) = C^x(\ell) e^{-\ell/\xi} [1 + o(\ell^0)]. \quad (22)$$

Here ξ and $C^x(\ell)$ are functions of the initial (h_0) and final (h) magnetic fields. The inverse correlation length is

$$\xi^{-1} = \theta_H(h-1)\theta_H(h_0-1) \ln[\min(h_0, h_1)] - \ln[x_+ + x_- + \theta_H((h-1)(h_0-1))\sqrt{4x_+x_-}], \quad (23)$$

where $\theta_H(x)$ is the Heaviside step function and

$$x_{\pm} = \frac{[\min(h, h^{-1}) \pm 1][\min(h_0, h_0^{-1}) \pm 1]}{4}, \quad h_1 = \frac{1 + hh_0 + \sqrt{(h^2 - 1)(h_0^2 - 1)}}{h + h_0}. \quad (24)$$

The function $C^x(\ell)$ describes the subleading large- ℓ asymptotics and takes the following form

(i) Quench within the ferromagnetic phase ($h_0, h < 1$).

$$C^x(\ell) = \frac{1 - hh_0 + \sqrt{(1-h^2)(1-h_0^2)}}{2\sqrt{1-hh_0}\sqrt[4]{1-h_0^2}} \equiv C_{\text{FF}}^x. \quad (25)$$

(ii) Quench from the ferromagnetic to the paramagnetic phase ($h_0 < 1 < h$).

$$C^x(\ell) = \sqrt{\frac{h\sqrt{1-h_0^2}}{h+h_0}} \equiv C_{\text{FP}}^x. \quad (26)$$

(iii) Quench from the paramagnetic to the ferromagnetic phase ($h_0 > 1 > h$).

$$C^x(\ell) = \sqrt{\frac{h_0-h}{\sqrt{h_0^2-1}}} \cos\left(\ell \arctan \frac{\sqrt{(1-h^2)(h_0^2-1)}}{1+h_0h}\right) \equiv C_{\text{PF}}^x(\ell). \quad (27)$$

(iv) Quench within the paramagnetic phase ($1 < h_0, h$).

$$C^x(\ell) \equiv C_{\text{PP}}^x(\ell) = \begin{cases} -\frac{h_0\sqrt{h}(hh_0-1+\sqrt{(h^2-1)(h_0^2-1)})^2}{4\sqrt{\pi}(h_0^2-1)^{3/4}(h_0h-1)^{3/2}(h-h_0)} \ell^{-3/2} & \text{if } 1 < h_0 < h, \\ \sqrt{\frac{h(h_0-h)\sqrt{h_0^2-1}}{(h+h_0)(hh_0-1)}} & \text{if } 1 < h < h_0. \end{cases} \quad (28)$$

Here $\arctan(|x|) \in [0, \pi/2)$. We see that in most cases $C_x(\ell)$ tends to a constant value at large ℓ . The exceptions are quenches from the paramagnetic to the ferromagnetic phase, where $C^x(\ell)$ tends to an oscillatory function with constant amplitude, and quenches to a larger magnetic field within the paramagnetic phase, where $C^x(\ell)$ decays like a power law with exponent $-3/2$. For the special cases $h_0 = 0$ and $h_0 = \infty$ the asymptotic behaviour of $\rho^{xx}(\ell \gg 1, t = \infty)$ agrees with the results of Ref. [55], which were obtained by different methods.

2.3. Transverse Correlators in the Stationary State

The stationary behaviour of the connected longitudinal two-point function is

$$\rho_c^{zz}(\ell \gg 1, t = \infty) \simeq C^z \ell^{-\alpha^z} e^{-\ell/\xi_z} (1 + O(\ell^{-1})), \quad (29)$$

where the transverse correlation length ξ_z , the exponent α^z and the amplitude C^z are given by

$$\xi_z^{-1} = |\ln h_0| + \min(|\ln h_0|, |\ln h|),$$

$$\alpha^z = \begin{cases} 1 & \text{if } |\ln h| > |\ln h_0|, \\ 0 & \text{if } h_0 = 1/h, \\ 1/2 & \text{if } |\ln h| < |\ln h_0|, \end{cases} \quad (30)$$

$$C^z = \begin{cases} \frac{|h_0-1/h_0|}{4\pi} \frac{h_0-h}{hh_0-1} & \text{if } |\ln h| > |\ln h_0|, \\ -\frac{(h-1/h)^2}{2\pi} & \text{if } h_0 = 1/h, \\ \frac{(h-1/h)\sqrt{|h_0-1/h_0|(h_0-h)}}{8\sqrt{\pi}h} \sqrt{\frac{h_0-h}{h_0(hh_0-1)}} \frac{e^{\text{sgn}(\ln h)|\ln h_0|/2}}{\sinh \frac{|\ln h|+|\ln h_0|}{2}} & \text{if } |\ln h| < |\ln h_0|. \end{cases} \quad (31)$$

2.4. How long one has to wait before one can “see” the GGE

Given that the infinite time behaviour of the reduced density matrix is described by a GGE, a natural question to ask is how long we need to wait in order to be able to detect the convergence of a given observable to its stationary value. Clearly, to answer this question we require knowledge of the full time evolution of correlation functions and not only their stationary values. This is very difficult in general, but can be studied in detail for the simple yet instructive case of the transverse spin-spin correlation function.

2.4.1. *Transverse Correlations* The connected transverse spin-spin correlation function has the following integral representation

$$\rho_c^{zz}(\ell, t) = \int_{-\pi}^{\pi} \frac{dk_1 dk_2}{(2\pi)^2} e^{i\ell(k_1 - k_2)} \left\{ \prod_{j=1}^2 \sin \Delta_{k_j} \sin(2\varepsilon_h(k_j)t) - \prod_{j=1}^2 e^{i\theta_{k_j}} \left[\cos \Delta_{k_j} - i \sin \Delta_{k_j} \cos(2\varepsilon_h(k_j)t) \right] \right\}. \quad (32)$$

The integral representation can now be used to obtain asymptotic expansions for large ℓ and t in the two limits of interest:

- (i) In the space-time scaling limit $\ell, v_{\max}t \rightarrow \infty$ with $v_{\max}t/\ell$ fixed, the asymptotic behaviour can be evaluated by means of a stationary phase approximation. This shows that the leading behaviour is a t^{-1} power-law decay, while subleading corrections are power laws as well, i.e.

$$\rho_c^{zz}(\ell = \frac{v_{\max}t}{\kappa}, t) \sim \frac{D^z(t)}{\kappa^2 t} + o(t^{-1}), \quad (33)$$

where $D^z(t)$ is sum of a constant contribution and oscillatory terms with constant amplitudes.

- (ii) In the late time regime at fixed, large ℓ , we find that $\rho_c^{zz}(\ell, t)$ decays as a power law in t to a stationary value that is exponentially small in ℓ

$$\rho_c^{zz}(\ell, t) \sim \rho_c^{zz}(\ell, \infty) + \frac{E^z(t)\ell e^{-\ell/\tilde{\xi}_z}}{t^{3/2}} + o(t^{-3/2}), \quad (34)$$

where $E^z(t) = \sum_{q=0, \pi} A_q \cos(2t\varepsilon_h(q) + \varphi_q)$ with constants A_q, φ_q , and the stationary value $\rho_c^{zz}(\ell, \infty)$ is given above in (29). Crucially, $\rho_c^{zz}(\ell, \infty) \propto e^{-\ell/\xi_z}$ is exponentially small in ℓ . The inverse correlation length $\tilde{\xi}_z^{-1}$ is given by

$$\tilde{\xi}_z^{-1} = \min(|\log h_0|, |\log h|) < \xi_z^{-1}. \quad (35)$$

We note that in the space-time scaling limit exponentially small terms such as $\rho_c^{zz}(\ell, \infty)$ will always be negligible compared to the dominant power law behaviour. On the other hand, in the late time regime there exists a cross-over time scale

$$Jt_2^* \sim e^{(2\ell/3)|\log h_0|}, \quad (36)$$

after which the stationary behaviour becomes apparent. Importantly, this time scale is *exponentially large* in the separation ℓ . As we have alluded to before, the space-time scaling limit is a convenient way of obtaining the behaviour of $\rho_c^{zz}(\ell, t)$ for general large ℓ and t . Contributions that are exponentially small in ℓ are not included in the asymptotic expansion obtained in the space-time scaling limit. This observation allows us to define a cross-over time scale t_1^* between the intermediate and late-time regimes by the requirement that the leading asymptotics in the space-time scaling limit ($\propto \ell^2/t^3$) becomes comparable to the neglected exponentially small terms $O(e^{-\ell/\xi_z}, e^{-\ell/\tilde{\xi}_z}/t^{3/2})$. This gives

$$Jt_1^* \sim e^{2\ell/3\tilde{\xi}_z}. \quad (37)$$

This estimate suggests that the result obtained in the space-time scaling limit will give a good description of $\rho_c^{zz}(\ell, t)$ for *fixed* separation ℓ up to times of order t_1^* . When $|\log h_0| < |\log h|$, the two time scales are comparable $t_1^* \sim t_2^*$. This means that in practice the space-time scaling limit provides a good approximation for $\rho_c^{zz}(\ell, t)$ all the way up to the stationary regime (see Fig. 2). On

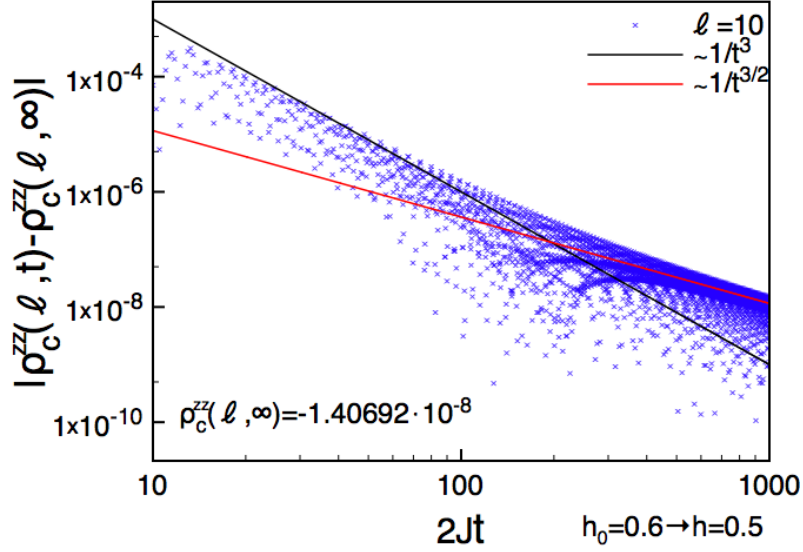


Figure 2. Absolute value of the connected transverse correlator $\rho_c^{zz}(\ell, t) - \rho_c^{zz}(\ell, t = \infty)$ as function of time at fixed $\ell = 10$. Initial and final magnetic fields $h_0 = 0.6 \rightarrow h = 0.5$ are chosen such that a crossover between a t^{-3} and the asymptotic $t^{-3/2}$ regime occurs at accessible times.

the other hand, when $|\log h_0| > |\log h|$ there is a large time window $t_2^* > t > t_1^*$ in which a $t^{-3/2}$ power-law decay can be observed.

The implications of these considerations for experiments or numerical calculations on finite-size systems are as follows:

- First and foremost, in order to observe the GGE either numerically or experimentally, it is essential to focus on a suitable observable. For the case of the TFIC this would be e.g. the one-point function $\rho^z(t)$.
- In a finite system of size L , boundary effects (such as reflection from the boundaries or completing a full system traverse in the periodic case) will become important at a timescale

$$t_{\text{fs}} \sim L/v_{\text{max}}. \quad (38)$$

In order to observe stationary behaviour, i.e. the GGE, this timescale should be larger than the cross-over scale t_2^*

$$t_{\text{fs}} > t_2^*. \quad (39)$$

In practice this requires extremely large system sizes. Just to quote some numbers: if we could simulate chains of length $L = 10000$ (which is at least one order of magnitude larger than what can be achieved with currently available numerical algorithms), the GGE can be “seen” at best for correlations evaluated at distances of about 10 lattice sites. Correlation functions evaluated at larger distances may *appear* stationary, but this behaviour will not be related to the GGE.

- The behaviour for large $v_{\text{max}}t < v_{\text{max}}t_{\text{fs}}$ and distances ℓ (compared to the lattice spacing) for a finite system will generically be described by the intermediate time regime and in particular

the result obtained in the space-time scaling limit. Time scales necessary to observe stationary behaviour will generically be inaccessible.

2.4.2. Longitudinal Correlations As a result of their non-local nature with respect to the Jordan-Wigner fermions these are much more difficult to determine than the transverse correlations. Detailed results for the time evolution of one and two-point functions of σ_m^x are presented in paper I. The implications of these results for the present discussion are summarized as follows.

- For quenches originating in the ferromagnetic phase, the stationary value of $\rho^{xx}(\ell, t)$ emerges after a time t_3^* that scales as a power-law with the spatial separation $v_{\max} t_3^* \sim \ell^{4/3}$, cf. Eq. (28) of paper I. Hence it is straightforward to see the approach (as a function of time) of the two-point function to its stationary values, which are given by the GGE.
- For quenches within the paramagnetic phase $\rho_c^{xx}(\ell, t)$ exhibits an oscillatory power-law decay in time towards its stationary value, which is exponentially small in ℓ . Hence, in complete analogy to the case of the transverse two-point function, the time scale t_2^* after which the stationary behaviour reveals itself, is exponentially large

$$t_2^* \propto e^{2\ell/3\xi}, \quad (40)$$

and very difficult to observe in practice.

2.5. Generalized Gibbs Ensemble and Conformal Field Theory.

In Refs. [11, 12] it has been shown that for conformal field theories the long time limit of correlation functions after a quantum quench described by a conformally invariant boundary state is described by a *thermal* (Gibbs) ensemble. Since CFTs are prototypical integrable field theories, this result apparently contradicts the general expectation that integrable models should be described by appropriately defined GGEs. We now show that for the case of the Ising field theory there is in fact no contradiction and that the peculiarity of the CFT result can be traced back to the particular (non-generic) initial state considered in Refs. [11, 12].

As discussed in paper I, CFT describes the scaling limit of the Ising model when the final gap is sent to zero. The scaling limit of the transverse field Ising chain is (a_0 is the lattice spacing)

$$J \rightarrow \infty, \quad h \rightarrow 1, \quad a_0 \rightarrow 0, \quad (41)$$

while keeping fixed both the gap Δ and the velocity v

$$2J|1 - h| = \Delta, \quad 2Ja_0 = v. \quad (42)$$

In this limit the dispersion and Bogoliubov angle become

$$\varepsilon(q) = \sqrt{\Delta^2 + v^2 q^2}, \quad \theta_h(q) \rightarrow \text{sgn}(h - 1) \arctan\left(\frac{vq}{\Delta}\right). \quad (43)$$

Here the physical momentum is defined as $q = k/a_0$, with $-\infty < q < \infty$. In our quench problem both the initial and the final magnetic field are scaled to the critical point, i.e. we need to take

$$h_0 \rightarrow 1, \quad 2J|1 - h_0| = \Delta_0 = \text{fixed}. \quad (44)$$

Thus in the scaling limit we have

$$\cos \Delta_k \rightarrow \frac{\Delta \Delta_0 + (vq)^2}{\sqrt{(\Delta^2 + (vq)^2)(\Delta_0^2 + (vq)^2)}}. \quad (45)$$

For quenches to the quantum critical point, which is described by the Ising conformal field theory with central charge $c = 1/2$, we need to furthermore take $\Delta \rightarrow 0$, which results in

$$\cos \Delta_k \rightarrow \frac{v|q|}{\sqrt{\Delta_0^2 + (vq)^2}}. \quad (46)$$

In this limit the expression (18) for the Lagrange multipliers β_k becomes

$$\beta_q v|q| = 2 \operatorname{arctanh} \frac{v|q|}{\sqrt{\Delta_0^2 + (vq)^2}}. \quad (47)$$

This shows that one obtains mode-dependent temperatures and hence a non-trivial GGE even in the case of a quench to the Ising CFT.

However, the particular boundary states considered as initial states in Refs. [11, 12] correspond to the limit of infinitesimal correlation length, i.e. a very large initial gap $\Delta_0 \rightarrow \infty$. In the limit $v|q|/\Delta_0 \rightarrow 0$, (47) becomes

$$\beta_q = \frac{2}{\Delta_0}, \quad (48)$$

i.e. all mode dependent temperatures *become equal*, resulting in a thermal state. Thus the findings [11, 12] should not be interpreted as showing that CFTs thermalize after quantum quenches. We have been informed by John Cardy that by perturbing the conformal boundary condition it is possible to show the emergence of a mode dependent temperature directly in CFT.

3. The infinite time limit and the generalized Gibbs ensemble

In this section we consider the infinite time limit of the reduced density matrix ρ_A of a subsystem A composed of ℓ contiguous spins. To analyze ρ_A , we first consider its building blocks, i.e. the two-point real-space correlation functions of fermions. It is convenient to replace the Jordan-Wigner fermions c_j (as defined in Appendix A of paper I) by the (real) Majorana fermions

$$a_j^x = c_j^\dagger + c_j \quad a_j^y = i(c_j^\dagger - c_j), \quad (49)$$

which satisfy the algebra $\{a_l^x, a_n^x\} = 2\delta_{ln}$, $\{a_l^y, a_n^y\} = 2\delta_{ln}$, $\{a_l^x, a_n^y\} = 0$. In terms of these Majorana fermions, the operator σ_j^x has the nonlocal representation

$$\sigma_\ell^x = \prod_{j=1}^{\ell-1} (ia_j^y a_j^x) a_\ell^x, \quad (50)$$

while σ_l^z is local

$$\sigma_l^z = ia_l^y a_l^x. \quad (51)$$

The *correlation matrix* Γ is defined as

$$\Gamma = \begin{bmatrix} \Gamma_0 & \Gamma_{-1} & \cdots & \Gamma_{1-\ell} \\ \Gamma_1 & \Gamma_0 & & \vdots \\ \vdots & & \ddots & \vdots \\ \Gamma_{\ell-1} & \cdots & \cdots & \Gamma_0 \end{bmatrix}, \quad \Gamma_l = \begin{pmatrix} -f_l & g_l \\ -g_{-l} & f_l \end{pmatrix}, \quad (52)$$

where the matrix elements are the fermionic correlators

$$\begin{aligned} g_n &\equiv -\langle a_l^x a_{l+n}^y \rangle, \\ f_n + \delta_{n0} &\equiv \langle a_l^x a_{l+n}^x \rangle = \langle a_l^y a_{l+n}^y \rangle \quad \forall l. \end{aligned} \quad (53)$$

The matrix Γ is a block Toeplitz matrix, because its constituent 2×2 blocks depend only on the difference between row and column indices. It is customary to introduce the (block) *symbol* of the matrix Γ as follows

$$\hat{\Gamma}(k) = \begin{pmatrix} -f(k) & g(k) \\ -g(-k) & f(k) \end{pmatrix} \equiv \sum_{n=-\infty}^{\infty} e^{-ink} \Gamma_n. \quad (54)$$

The functions $f(k)$ and $g(k)$ are

$$\begin{aligned} f(k) &= \sin \Delta_k \sin(2\varepsilon_h(k)t), \\ g(k) &= -ie^{i\theta_k} \left[\cos \Delta_k - i \sin \Delta_k \cos(2\varepsilon_h(k)t) \right], \end{aligned} \quad (55)$$

where $e^{i\theta_k}$ and $\cos \Delta_k$ are defined in (10) and (11) respectively, while

$$\sin \Delta_k = \frac{(h - h_0) \sin k}{\sqrt{1 + h^2 - 2h \cos k} \sqrt{1 + h_0^2 - 2h_0 \cos k}}. \quad (56)$$

The infinite time limit of the fermion correlators (53) is straightforwardly obtained by Fourier transforming the functions $f(k)$ and $g(k)$

$$\begin{aligned} f_j^\infty &= \langle a_l^x a_{l+j}^x \rangle \Big|_{t=\infty} - \delta_{j0} = 0, \\ g_j^\infty &= -\langle a_l^x a_{l+j}^y \rangle \Big|_{t=\infty} = -i \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ijk} e^{-i\theta_k} \cos \Delta_k. \end{aligned} \quad (57)$$

We can now use the Wick theorem to construct all correlation functions in the Ising chain[‡]. Moreover, as shown in Refs [66, 67], the matrix Γ determines the full reduced density matrix of the block A of ℓ contiguous fermions (and hence spins, because contiguous spins are mapped to contiguous fermions, see Eq. (50)) in the chain

$$\rho_A = \frac{1}{2^\ell} \sum_{\mu_l=0,1} \left\langle \prod_{l=1}^{2\ell} a_l^{\mu_l} \right\rangle \left(\prod_{l=1}^{2\ell} a_l^{\mu_l} \right)^\dagger \propto e^{a_l W_{lm} a_m / 4}, \quad (58)$$

where $a_{2n} = a_n^x$, $a_{2n-1} = a_n^y$ and

$$\tanh \frac{W}{2} = \Gamma. \quad (59)$$

Given ρ_A one can calculate any local correlation function with support in A .

In order to understand the properties of the stationary reduced density matrix, we should study the structure of the fermion correlations in Eq. (57). These resemble the analogous correlations of an Ising chain in equilibrium at finite temperature β^{-1} calculated in Ref. [52]

$$\begin{aligned} f_j^{(\beta)} &= \langle\langle a_l^x a_{l+j}^x \rangle\rangle - \delta_{j0} = 0, \\ g_j^\beta &= -\langle\langle a_l^x a_{l+j}^y \rangle\rangle = -i \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ijk} e^{-i\theta_k} \tanh\left(\frac{\beta \varepsilon_k}{2}\right), \end{aligned} \quad (60)$$

[‡] For $t \rightarrow \infty$, the expectation values of odd (with respect to fermion parity) operators vanish for any quench. Thus all nonzero correlation functions can be written in terms of two-point fermion correlators only.

where $\langle\langle \mathcal{O} \rangle\rangle$ denotes the equilibrium expectation value of the operator \mathcal{O} at temperature $1/\beta$. Comparing (60) to (57) suggests that if it were possible to find a β such that $\cos \Delta_k = \tanh(\beta \varepsilon_k/2)$ for any k , local properties of the (sub)system could be described by a *thermal state* at inverse temperature β^{-1} . It is easy to see that this is not possible: on the one hand, the dispersion relation is a continuous bounded function such that $\tanh(\beta \varepsilon_k/2) < 1$ for any finite β . On the other hand we have $|\cos \Delta_0| = |\cos \Delta_\pi| = 1$, which implies that the equality cannot be fulfilled at least in some neighbourhood of the momenta 0 and π .

Having ruled out the possibility of a thermal reduced density matrix in the stationary state, we next investigate the conjecture put forward by Rigol et. al. [9], according to which the stationary state of integrable models are described by a *generalized Gibbs ensemble*. The latter is obtained by maximizing the entropy, while keeping the energy as well as all higher conservation laws fixed. The resulting stationary reduced density matrix is of the form

$$\rho_{\text{GGE}} = \frac{1}{Z_{\text{GGE}}} e^{-\sum_m \lambda_m I_m}, \quad (61)$$

where the I_m represent a complete set of independent, local integrals of motion and the Lagrange multipliers λ_m are fully determined by the initial state $|\Psi_0\rangle$ through the conditions

$$\text{Tr}[\rho_{\text{GGE}} I_m] = \langle \Psi_0 | I_m | \Psi_0 \rangle. \quad (62)$$

We stress that locality of I_m is an essential feature of this formulation. For the case of the Ising model, it is shown in Ref. [64] that the local integrals of motion can be expressed as *linear* combinations of the Bogoliubov fermion number operators $n_k \equiv \alpha_k^\dagger \alpha_k$. Hence for the Ising model (and by analogy other theories with free fermionic spectra) one may replace the local charges I_m in (61) by the number operators n_k , despite the fact that the latter are non-local as discussed in the introduction. Thus, taking the Lagrange multipliers as $\lambda_k = \beta_k \varepsilon_k$ in order to facilitate the eventual identification of β_k as a mode-dependent temperature, the GGE takes the form

$$\rho_{\text{GGE}} = \frac{1}{Z_{\text{GGE}}} e^{-\sum_k \beta_k \varepsilon_k \alpha_k^\dagger \alpha_k}. \quad (63)$$

The Lagrange multipliers β_k are fixed by the conditions

$$\text{Tr}[\rho_{\text{GGE}} n_k] = \langle \Psi_0 | n_k | \Psi_0 \rangle, \quad (64)$$

which can be solved in closed form by

$$\beta_k \varepsilon_k = 2 \arctanh(\cos \Delta_k). \quad (65)$$

The fermionic two-point correlators in the GGE can be straightforwardly evaluated following the finite-temperature calculation and are given by

$$g_j^{(\text{GGE})} = -i \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ij k} e^{-i\theta_k} \tanh\left(\frac{\beta_k \varepsilon_k}{2}\right). \quad (66)$$

The crucial point now is that the fermionic two-point functions g_j^∞ and $g_j^{(\text{GGE})}$ are exactly the same, given that the Lagrange multipliers β_k are related to Δ_k by (65), i.e.

$$g_j^\infty = g_j^{(\text{GGE})}. \quad (67)$$

As the fermion correlators f_j^∞ and g_j^∞ completely fix the reduced density matrix ρ_A (the matrix W_{lm} in (58) can be determined by simply evaluating all fermionic two-point functions using Wick's theorem), the equality (67) is *lifted to the level of the reduced density matrices*

$$\rho_A = \text{Tr}_A(\rho_{\text{GGE}}). \quad (68)$$

A characteristic feature of the GGE is that it retains detailed information about the initial state. This is in contrast to a thermal ensemble, which would correspond to $\beta_k = \beta$ for any k and would depend on the initial state only through the average value of the energy. Given the structure of Δ_k there is no quench for which the β_k is independent on k and hence the stationary state is never thermal. However, it is possible for the differences between GGE and thermal ensemble to be small for certain observables [49].

To summarize the results of this section, we have shown that the stationary reduced density matrix after an arbitrary quench of the bulk magnetic field in the Ising chain is equivalent to a generalized Gibbs ensemble. Hence all local correlation functions can be deduced from the GGE. The calculation of observables in the framework of the GGE generically remains a difficult problem. In the following sections we determine the stationary behaviour of some of the most important observables, namely the longitudinal and transverse correlation functions.

4. Stationary value of the longitudinal correlation function

The two-point function of σ_j^x is the expectation value of a string of Majorana fermions

$$\rho^{xx}(\ell, t) = \langle \prod_{j=1}^{\ell} (-i a_j^y(t) a_{j+1}^x(t)) \rangle, \quad (69)$$

and by Wick's theorem one obtains a representation as the Pfaffian of the $2\ell \times 2\ell$ antisymmetric matrix $\bar{\Gamma}$ considered in Ref. [52]

$$\rho^{xx}(\ell, t) = \text{pf}(\bar{\Gamma}), \quad (70)$$

which has the same structure of the correlation matrix in Eq. (52) with blocks

$$\bar{\Gamma}_l = \begin{pmatrix} -if_l & -ig_{l-1} \\ ig_{-l-1} & if_l \end{pmatrix}. \quad (71)$$

From the asymptotic correlations (57), we can compute this correlation function in the limit of infinite time. Since $f_n^\infty = 0$, the block structure in the matrix $\bar{\Gamma}$ simplifies, and the two-point function of the order parameter in the infinite time limit becomes the determinant of a $\ell \times \ell$ Toeplitz matrix G

$$\rho^{xx}(\ell, t) = \det(G), \quad G_{ln} = g_{l-n}^\infty. \quad (72)$$

The symbol of G is

$$g_\infty(k) = -e^{i(k-\theta_k)} \cos \Delta_k. \quad (73)$$

The representation (72) is more convenient than (70) for determining the large- ℓ asymptotics of $\rho^{xx}(\ell, t) = \det(G)$. We will show that at late times the two-point function of the order parameter decays exponentially with the distance

$$\rho^{xx}(\ell \gg 1, t = \infty) = C^x(\ell) e^{-\ell/\xi} [1 + o(\ell^0)]. \quad (74)$$

The result (74) is obtained by using exact results on the asymptotic behaviour of the determinant of Toeplitz matrices, such as Szëgo's lemma, the Fisher-Hartwig conjecture, and generalizations thereof (the results we need in the following are summarized in Appendix A and are taken from Refs. [68, 69, 70, 71]). To apply these theorems it is useful to change variable as $z = e^{ik}$, so that the symbol can be rewritten as

$$\mathfrak{g}_\infty(z) \equiv g_\infty(-i \ln z) = \frac{h + h_0}{2\sqrt{h_0}} \frac{\sqrt{z}}{\sqrt{h_0 - z} \sqrt{z - 1/h_0}} \frac{(z - h_1)(1/h_1 - z)}{z - h}, \quad (75)$$

where h_1 is defined in Eq. (24). The asymptotic behaviour of the determinant and so the physical quantities ξ and $C^x(\ell)$ depend on the analytic properties of the symbol, which in turn change with the quench parameters.

4.1. Quenches within the ferromagnetic phase.

The symbol (75) is different from zero on the unit circle $|z| = 1$ and has winding number zero about the origin $z = 0$. Under these conditions, it is possible to apply the strong Szëgo lemma (see (A.3)). The inverse correlation length is straightforwardly obtained as

$$\xi^{-1} = - \int_0^{2\pi} \frac{dk}{2\pi} \ln \mathfrak{g}_\infty(e^{ik}) = - \int_0^{2\pi} \frac{dk}{2\pi} \ln \cos \Delta_k. \quad (76)$$

The integral can be carried out using ($x > 0$)

$$\int_0^{2\pi} \frac{dk}{2\pi} \ln(x - e^{ik}) = \theta(x - 1) \ln x, \quad (77)$$

to finally obtain

$$\xi^{-1} = - \ln\left(\frac{h + h_0}{2} h_1\right). \quad (78)$$

The multiplicative factor $C^x(\ell) = \mathcal{C}_{\text{FF}}^x$ is slightly more complicated to obtain. It is given by (see Eq. (A.3))

$$\mathcal{C}_{\text{FF}}^x = \exp\left[\sum_{k \geq 1} k(\ln \mathfrak{g}_\infty)_k (\ln \mathfrak{g}_\infty)_{-k}\right], \quad (79)$$

where we used the same notation as in the appendix. The Fourier coefficients $(\ln \mathfrak{g}_\infty)_k$ are

$$(\ln \mathfrak{g}_\infty)_k = \begin{cases} (h_0^k - 2h_1^{-k})/(2k) & \text{if } k > 0, \\ \ln((h + h_0)h_1/2) & \text{if } k = 0, \\ (2h_1^k - 2h^{-k} - h_0^{-k})/(2k) & \text{if } k < 0. \end{cases} \quad (80)$$

Carrying out the k -sum in (79) we obtain

$$\mathcal{C}_{\text{FF}}^x = \frac{1 - hh_0 + \sqrt{(1 - h^2)(1 - h_0^2)}}{2\sqrt{1 - hh_0} \sqrt[4]{1 - h_0^2}}. \quad (81)$$

Putting everything together we conclude that the two-point function has the following asymptotic behaviour

$$\rho^{xx}(\ell, t = \infty) = \frac{1 - hh_0 + \sqrt{(1 - h^2)(1 - h_0^2)}}{2\sqrt{1 - hh_0} \sqrt[4]{1 - h_0^2}} e^{-\ell/\xi} + O(e^{-\kappa\ell}), \quad (82)$$

for some $\kappa > \xi^{-1}$. In Fig. 3 this analytic prediction is compared with numerical data. The agreement is clearly excellent.

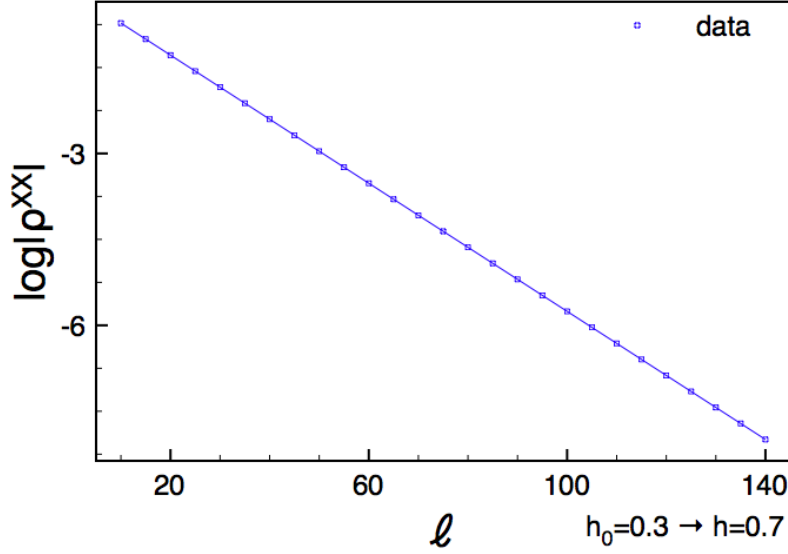


Figure 3. $\ln |\rho^{xx}(\ell, t = \infty)|$ after a quench within the ordered phase from $h_0 = 0.3$ to $h = 0.7$. The straight line is the asymptotic prediction in Eq. (82).

4.2. Quenches within the paramagnetic phase.

Here the winding number of the symbol equals 1. Following Ref. [71] (see Appendix A for some details) we first rewrite the symbol by factorizing an appropriate power of z , such that the remainder has winding number zero about the origin

$$y(z) \equiv -\mathfrak{g}_\infty(1/z)z = \frac{h + h_0}{2h\sqrt{h_0}} \frac{\sqrt{z}}{\sqrt{z - 1/h_0}\sqrt{h_0 - z}} \frac{(z - 1/h_1)(h_1 - z)}{z - 1/h}. \quad (83)$$

The function $y(z)$ has winding number zero as desired and no zeroes on the unit circle $|z| = 1$. We note that $\mathfrak{g}_\infty(1/z)$ is in fact the symbol of the transpose of G . It has negative winding number, which is precisely the case discussed in Appendix A. From the Wiener-Hopf factorization of $y(z)$ as defined in Eq. (A.9)

$$\begin{aligned} y(z) &= y_+(z) \frac{h + h_0}{2hh_0} h_1 y_-(z), \\ y_\pm(z) &= \exp \sum_{k \geq 1} (\ln y)_{\pm k} z^{\pm k}, \end{aligned} \quad (84)$$

the correlation function is obtained using (A.16)

$$\rho^{xx}(\ell, t = \infty) \simeq \exp \left[\sum_{k \geq 1} k (\ln y)_k (\ln y)_{-k} \right] \left(\frac{h + h_0}{2hh_0} h_1 \right)^\ell I_\ell. \quad (85)$$

Here I_ℓ is given in terms of $y_\pm(z)$ by

$$I_\ell = \int_0^{2\pi} \frac{dk}{2\pi} e^{i\ell k} \frac{y_-(e^{-ik})}{y_+(e^{-ik})}. \quad (86)$$

The inverse correlation length can be read off from (85) to be

$$\xi^{-1} = -\ln\left(\frac{h+h_0}{2hh_0}h_1\right) - \lim_{\ell \rightarrow \infty} \frac{1}{\ell} \ln I_\ell. \quad (87)$$

In order to find $y_-(z)$ we use the standard Wiener-Hopf factorizations

$$(\alpha(x-z)^q)_- = \begin{cases} \left(\frac{z-x}{z-1}\right)^q & \text{if } x < 1, \\ 1 & \text{if } x > 1, \end{cases} \quad (88)$$

$$(\alpha(x-z)^q)_+ = \begin{cases} (1-z)^q & \text{if } x < 1, \\ (1-z/x)^q & \text{if } x > 1, \end{cases} \quad (89)$$

where α is an arbitrary complex number. The rhs above are independent on α because of our normalization of the factorization in Eq. (A.9).

The Wiener-Hopf factorization of $y(z)$ is a combination of the above with $q = 0, \pm 1/2, \pm 1$, and after some simple algebra we get

$$y_-(z) = \frac{\sqrt{z}}{\sqrt{z-1/h_0}} \frac{z-1/h_1}{z-1/h}, \quad y_+(z) = \frac{\sqrt{h_0}}{h_1} \frac{h_1-z}{\sqrt{h_0-z}}. \quad (90)$$

Substituting these expressions into I_ℓ , we obtain

$$I_\ell = \frac{h\sqrt{h_0}}{h_1} \oint_C \frac{dz}{2\pi i} z^{\ell-\frac{1}{2}} \frac{\sqrt{z-1/h_0}}{\sqrt{h_0-z}} \frac{h_1-z}{(h-z)(z-1/h_1)}, \quad (91)$$

where C is the unit circle. For numerical computations the following equivalent expression is more convenient

$$I_\ell = 2Jh \int_0^{2\pi} \frac{dk}{2\pi} \frac{e^{i\ell k}}{\varepsilon_h(k)} e^{-i(\theta(k)+\theta_0(k)-2\theta_1(k))}, \quad (92)$$

where θ_1 is the Bogoliubov angle corresponding to the magnetic field h_1 , i.e.

$$e^{i\theta_1(k)} = \frac{h_1 - e^{ik}}{\sqrt{1 + h_1^2 - 2h_1 \cos k}}. \quad (93)$$

The integral (91) is dominated by the vicinities of the points in the interior of the unit circle, at which the integrand is non-analytic. There are two branch points at $z = 0$ and $z = 1/h_0$, and a simple pole at $z = 1/h_1$. If the leading contribution arises from $z \approx 1/h_1$, the ℓ -dependence takes a simple exponential form, while additional power-law corrections arise if the integral is dominated by the region $z \approx 1/h_0$. The leading exponential contribution arises from the pole if $h_0 > h_1$ (corresponding to $h_0 > h$) and by the branch point at $z = 1/h_0$ if $h_0 < h_1$ (corresponding to $h > h_0$)

$$I_\ell \propto \begin{cases} h_1^{-\ell} & h_0 > h, \\ h_0^{-\ell} & h_0 < h. \end{cases} \quad (94)$$

Combining this result with (86) and Eq. (87) leads to the following result for the inverse correlation length

$$\xi^{-1} = -\ln\left(\frac{h+h_0}{2hh_0}h_1\right) + \ln \min(h_0, h_1). \quad (95)$$

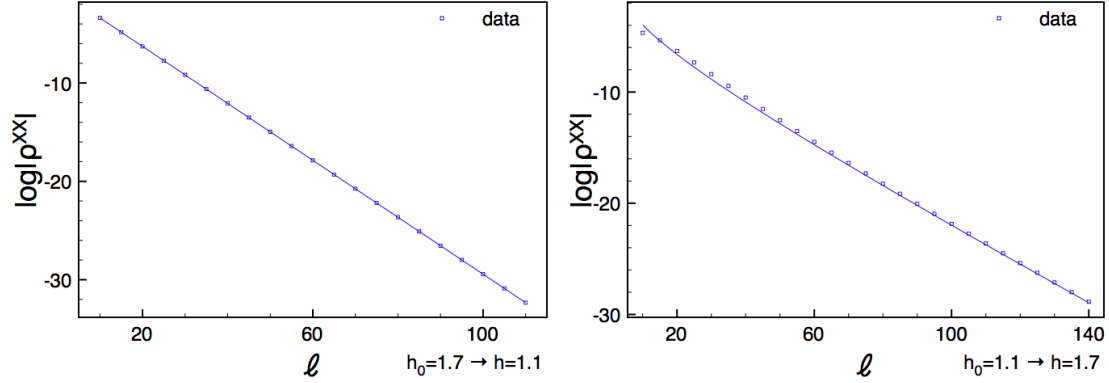


Figure 4. $\ln |\rho^{xx}(\ell, t = \infty)|$ as function of ℓ after two quenches within the disordered phase. Left: The case $h_0 > h$: The solid line is the prediction (101) and is seen to be in excellent agreement with the numerical data. Right: The case $h_0 < h$. The straight line is the leading term of the prediction (106). Some deviations at small ℓ are visible due to the subleading correction in (106).

4.2.1. *Calculation of the constant $\mathcal{C}_{\text{PP}}^x$ for $h_0 > h$.* From (85) we have

$$\mathcal{C}_{\text{PP}}^x = B \exp \left[\sum_{k \geq 1} k (\ln y)_k (\ln y)_{-k} \right], \quad (96)$$

where B is the residue of the integrand in Eq. (91) at $1/h_1$ without the exponential factor $h_1^{-\ell}$

$$B = \frac{2h\sqrt{h_0^2 - 1}}{(h_0 - h)\sqrt{h_0^2 - h^2}} \left(hh_0 - 1 - \sqrt{(h^2 - 1)(h_0^2 - 1)} \right). \quad (97)$$

The Fourier coefficients $(\ln y)_k$ are the same as for the quench from $1/h_0$ to $1/h$, because $\cos \Delta_k$ is invariant under the inversion of the magnetic fields, so that

$$(\ln y)_k \Big|_{h_0 \rightarrow h} = (\ln \mathbf{g}_{\infty}^{\text{FF}})_k \Big|_{1/h_0 \rightarrow 1/h}, \quad k \neq 0. \quad (98)$$

Using this in (96) we conclude that the “Szégo part” of $\mathcal{C}_{\text{PP}}^x$ is the same as in the ordered phase

$$\exp \left[\sum_{k \geq 1} k (\ln y)_k (\ln y)_{-k} \right] = \mathcal{C}_{\text{FF}}(1/h_0, 1/h). \quad (99)$$

Combining the two pieces in Eq. (96) we have

$$\mathcal{C}_{\text{PP}}^x = \sqrt{\frac{h(h_0 - h)\sqrt{h_0^2 - 1}}{(h + h_0)(hh_0 - 1)}}, \quad h_0 > h, \quad (100)$$

so that the two-point function has the following asymptotic behaviour

$$\rho^{xx}(\ell, t = \infty) = \sqrt{\frac{h(h_0 - h)\sqrt{h_0^2 - 1}}{(h + h_0)(hh_0 - 1)}} e^{-\ell/\xi} + O(e^{-\tilde{\kappa}\ell}), \quad h_0 > h, \quad (101)$$

where $\tilde{\kappa} > \xi^{-1}$. The asymptotic result (101) is compared to numerical results for the two-point function in Fig. 4 (left panel). The agreement is clearly excellent.

4.2.2. *Calculation of $\mathcal{C}_{\text{PP}}^x(\ell)$ for $h_0 < h$.* This case is slightly more involved. The two point function acquires a power law prefactor since the leading contribution to the integral (91) arises from the vicinity of the branch point at $1/h_0$. Isolating this contribution gives

$$I_\ell \approx -\frac{2h}{h_1} h_0^{-\frac{1}{2}-\ell} \int_{\frac{h_0}{h_1} + \epsilon_1 + i\epsilon}^{1+i\epsilon} \frac{dx}{2\pi i} \frac{x^{\ell-\frac{1}{2}} \sqrt{x-1}}{\sqrt{h_0-x/h_0}} \frac{h_1-x/h_0}{(h-x/h_0)(x/h_0-1/h_1)} + O(h_1^{-\ell}), \quad (102)$$

where ϵ and ϵ_1 are two infinitesimal positive constants. After the change of variable $w = (1-x)\ell_s$, with $\ell_s = \ell - \frac{1}{2}$, we obtain

$$I_\ell \approx -\frac{2h}{h_1} \frac{h_0^{-1-\ell_s}}{\ell_s^{\frac{3}{2}}} \int_0^{M_\ell} \frac{dw}{2\pi} \frac{\sqrt{w}(1-\frac{w}{\ell_s})^{\ell_s}}{\sqrt{h_0-\frac{1}{h_0}+\frac{w}{\ell_s h_0}}} \frac{h_1-\frac{1}{h_0}+\frac{w}{\ell_s h_0}}{(h-\frac{1}{h_0}+\frac{w}{\ell_s h_0})(\frac{1}{h_0}-\frac{1}{h_1}-\frac{w}{\ell_s h_0})}, \quad (103)$$

where

$$M_\ell = \ell_s \left(1 - \frac{h_0}{h_1} - \epsilon_1\right). \quad (104)$$

The large ℓ asymptotics is obtained by expanding in powers of $1/\ell_s$ and taking the integration limit (M_ℓ) to infinity. At a given order of the asymptotic expansion the resulting error is exponentially small. After straightforward algebra we arrive at

$$I_\ell \approx -\frac{\Gamma(3/2)}{2\pi} \frac{2hh_0}{\sqrt{h_0^2-1}} \frac{h_0 h_1 - 1}{(h_0 h - 1)(h_1 - h_0)} \frac{h_0^{-\ell}}{(\ell - \frac{1}{2})^{\frac{3}{2}}}, \quad (105)$$

which in turn leads to the following result for the two-point function

$$\begin{aligned} \rho^{xx}(\ell, t = \infty) \simeq & -\frac{h_0 \sqrt{h} \left(h h_0 - 1 + \sqrt{(h^2-1)(h_0^2-1)} \right)^2}{4\sqrt{\pi} (h_0^2-1)^{3/4} (h_0 h - 1)^{3/2} (h - h_0)} \\ & \times \left(\ell - \frac{1}{2} \right)^{-\frac{3}{2}} \left(1 + \frac{\alpha}{\ell - \frac{1}{2}} + \dots \right) e^{-\ell/\xi}, \quad h_0 < h. \end{aligned} \quad (106)$$

The constant α characterizing the subleading contribution in (106) is given by

$$\alpha = \frac{3}{8} \left[\frac{3}{h_0^2-1} - \frac{4}{h_0 h - 1} - \frac{h_0(5h_0(h-h_0) - 8\sqrt{(h^2-1)(h_0^2-1)})}{(h-h_0)(h_0^2-1)} \right]. \quad (107)$$

The requirement that the term involving α should be small establishes the regime of validity of the expansion (106). In particular, for small quenches in the sense of paper I, i.e. $\eta = \max |\sin \Delta_k|$ being small, we have

$$\alpha = \frac{3h}{2(h^2-1)} \frac{1}{\eta} + O(\eta^0). \quad (108)$$

Hence (106) holds only for distances that are sufficiently large compared to $(h-h_0)^{-1}$. In particular, in the limit $h \rightarrow h_0$ its region of validity disappears entirely.

In Fig. 4 the asymptotic result (106) (with α set to zero) is compared to numerical data. It is evident that for $h_0 < h$ and ℓ not large enough there are sizeable corrections to (106).

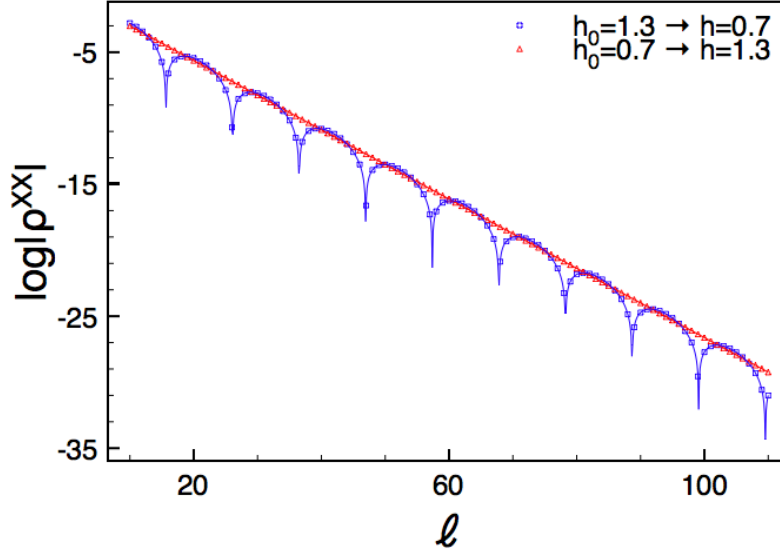


Figure 5. $\ln |\rho^{xx}(\ell, t = \infty)|$ as a function of ℓ for quenches across the critical point. Blue squares correspond to a quench from the paramagnetic ($h_0 = 1.3$) to the ferromagnetic ($h = 0.7$) phase. The blue solid line is the asymptotic expression (118). Red triangles correspond to a quench from the ferromagnetic ($h_0 = 0.7$) to the paramagnetic ($h = 1.3$) phase. The red straight line is the asymptotic result (116). In both cases the asymptotic and numerical results are seen to be in excellent agreement.

4.3. Quenches across the critical point.

For quenches across the critical point h_1 is a complex number with unit modulus $|h_1| = 1$. In this case, the symbol has two simple zeroes on the integration contour at $z_1 = h_1$ and $z_2 = 1/h_1 = h_1^*$. Thus, as first step we need to cast the symbol in the form (A.7) which is suitable for application of the Fisher-Hartwig conjecture as reported in (A.10). Since the symbol has no discontinuities, we only need to isolate the zeroes. This can be done by rewriting $g_\infty(z)$ in the form

$$\mathfrak{g}_\infty(z) = \left[h_1^{-\beta_1} (h_1^*)^{-\beta_2} \right] f^{(\beta_1, \beta_2)}(z) z^{\beta_1 + \beta_2} (-1)^{\beta_1 - \beta_2} \frac{(h_1 - z)(z - h_1^*)}{z}, \quad (109)$$

where

$$f^{(\beta_1, \beta_2)}(z) = \left[h_1^{\beta_1} (h_1^*)^{\beta_2} \frac{h + h_0}{2\sqrt{h_0}} \right] (-1)^{\beta_1 - \beta_2} \frac{\sqrt{z}}{\sqrt{h_0 - z} \sqrt{z - h_0^{-1}}} \frac{z^{1 - \beta_1 - \beta_2}}{z - h}. \quad (110)$$

The symbol (109) is now in the desired form (A.7) with

$$R = 2, \quad \alpha_0 = \beta_0 = 0, \quad \alpha_1 = \frac{1}{2}, \quad \beta_1 = \frac{1}{2} - m_1, \quad \alpha_2 = \frac{1}{2}, \quad \beta_2 = \frac{1}{2} - m_2, \quad (111)$$

where $m_{1,2} \in \mathbb{Z}$ are integers that label the inequivalent representations of the symbol. Taking into account the contributions of all representations results in an expression for the two-point function of the form

$$\rho^{xx}(\ell \gg 1, t = \infty) \simeq \sum_{\{\beta_1, \beta_2\}_{\text{ineq.}}} C_{\beta_1, \beta_2} \ell^{1/2 - \beta_1^2 - \beta_2^2} \left[-h_1^{\beta_1} (h_1^*)^{\beta_2} \right]^\ell e^{-\ell/\xi(\beta_1, \beta_2)}, \quad (112)$$

where we isolated the oscillatory factor $[-h_1^{\beta_1}(h_1^*)^{\beta_2}]^\ell$. In (112), $\xi(\beta_1, \beta_2)$ and C_{β_1, β_2} are respectively the correlation length and correlation amplitude corresponding to the representation labelled by $\beta_{1,2}$ of the symbol (*cf.* (A.10)). In the special case $R=2$, $\alpha_1 = \alpha_2 = 1/2$, $\alpha_0 = \beta_0 = 0$, $z_1 = h_1$, $z_2 = h_1^*$, the amplitudes of the various representations are given by

$$\begin{aligned} C_{\beta_1, \beta_2} = & \exp \left[\sum_{q \geq 1} q (\ln f^{(\beta_1, \beta_2)})_q (\ln f^{(\beta_1, \beta_2)})_{-q} \right] \left[f_+^{(\beta_1, \beta_2)}(h_1) \right]^{-\frac{1}{2} + \beta_1} \left[f_-^{(\beta_1, \beta_2)}(h_1) \right]^{-\frac{1}{2} - \beta_1} \\ & \times \left[f_+^{(\beta_1, \beta_2)}(h_1^*) \right]^{-\frac{1}{2} + \beta_2} \left[f_-^{(\beta_1, \beta_2)}(h_1^*) \right]^{-\frac{1}{2} - \beta_2} \\ & \times |h_1 - h_1^*|^{2(\beta_1 \beta_2 - \frac{1}{4})} \left(\frac{h_1^*}{h_1 e^{i\pi}} \right)^{\frac{\beta_2}{2} - \frac{\beta_1}{2}} \prod_{j=0}^2 \frac{G(\frac{3}{2} + \beta_j) G(\frac{3}{2} - \beta_j)}{G(2)}, \end{aligned} \quad (113)$$

where G is Barnes' G-function and $f_\pm^{(\beta_1, \beta_2)}(z)$ are the factors of the Wiener-Hopf factorization of $f^{(\beta_1, \beta_2)}(z)$ in the conventions (A.9). As $G(-n) = 0$ for $n = 0, 1, 2, \dots$ we need to consider only the four inequivalent representations

$$\beta_1 = \pm \frac{1}{2}, \quad \beta_2 = \pm \frac{1}{2}. \quad (114)$$

In order to obtain explicit expressions for $C_{\pm \frac{1}{2}, \pm \frac{1}{2}}$ and $\xi(\pm \frac{1}{2}, \pm \frac{1}{2})$ we need to consider quenches from the paramagnetic to the ferromagnetic phase and vice versa separately. This is done in the following two subsections.

4.3.1. Quenches from the paramagnetic to the ferromagnetic phase. The leading contributions in the large- ℓ limit arise from the two representations

$$(\beta_1, \beta_2) = \left(\frac{1}{2}, -\frac{1}{2} \right), \quad (\beta_1, \beta_2) = \left(-\frac{1}{2}, \frac{1}{2} \right). \quad (115)$$

In both cases the inverse correlation length takes the simple form

$$\xi^{-1} \left(\pm \frac{1}{2}, \mp \frac{1}{2} \right) = - \int_0^{2\pi} \frac{dk}{2\pi} \ln \left(h_1^{\mp 1} f^{(\pm \frac{1}{2}, \mp \frac{1}{2})}(e^{ik}) \right) = - \ln \left(\frac{h + h_0}{2h_0} \right). \quad (116)$$

As shown below, the constants C_{β_1, β_2} are equal to

$$C_{\frac{1}{2}, -\frac{1}{2}} = C_{-\frac{1}{2}, \frac{1}{2}} = \frac{1}{2} \sqrt{\frac{h_0 - h}{h_0^2 - 1}}. \quad (117)$$

Putting everything together we arrive at the following result for the asymptotics of the two-point function

$$\rho^{xx}(\ell, t = \infty) = \sqrt{\frac{h_0 - h}{h_0^2 - 1}} \operatorname{Re}[h_1^\ell] e^{-\ell/\xi} + O(e^{-\kappa' \ell}), \quad (118)$$

where $\kappa' > \xi^{-1}$. Rewriting h_1 as a function of h and h_0 one recovers the form given in (27). The expression (118) is compared to numerical results for $\rho^{xx}(\ell, t = \infty)$ in Fig. 5. The agreement for large distances ℓ is clearly excellent. We note that (118) can be understood as the leading term in an asymptotic expansion only if $\operatorname{Re}[h_1^\ell]$ is of order 1. For the particular quench from

$$h_0 = \frac{h + \sqrt{2}(1 - h^2)}{1 - 2h^2} \quad (119)$$

to $h < 1/\sqrt{2}$ we have $h_1^2 = i$ and (118) vanishes for $\ell = 4n + 2$, with n integer. For this particular sequence of distances ξ does then not represent the correlation length.

Calculation of the constant $C_{\frac{1}{2}, -\frac{1}{2}}$. The Wiener-Hopf factorization of $f^{(\frac{1}{2}, -\frac{1}{2})}(z)$ is

$$f^{(\frac{1}{2}, -\frac{1}{2})}(z) = f_+^{(\frac{1}{2}, -\frac{1}{2})}(z) \frac{(h + h_0)h_1}{2h_0} f_-^{(\frac{1}{2}, -\frac{1}{2})}(z), \quad (120)$$

where

$$f_+^{(\frac{1}{2}, -\frac{1}{2})}(z) = \frac{\sqrt{h_0}}{\sqrt{h_0 - z}}, \quad f_-^{(\frac{1}{2}, -\frac{1}{2})}(z) = \frac{\sqrt{z}}{\sqrt{z - 1/h_0}} \frac{z}{z - h}. \quad (121)$$

This results in Fourier coefficients

$$(\ln f^{(\frac{1}{2}, -\frac{1}{2})})_k = \begin{cases} h_0^{-k}/(2k) & \text{if } k > 0, \\ \ln\left(\frac{h+h_0}{2h_0} h_1\right) & \text{if } k = 0, \\ -(h_0^k + 2h^{-k})/(2k) & \text{if } k < 0. \end{cases} \quad (122)$$

The Szëgo part of the constant is then of the form

$$\exp\left[\sum_{k \geq 1} k (\ln f^{(\frac{1}{2}, -\frac{1}{2})})_k (\ln f^{(\frac{1}{2}, -\frac{1}{2})})_{-k}\right] = \frac{h_0}{\sqrt[4]{h_0^2 - 1} \sqrt{h_0 - h}}. \quad (123)$$

Substituting (121) and (123) into the expression (113) we obtain the result (117). The calculation of $C_{-\frac{1}{2}, \frac{1}{2}}$ is completely analogous.

4.3.2. Quenches from the ordered to the disordered phase. Here the leading contribution in (112) arises from the representation

$$\beta_1 = \beta_2 = \frac{1}{2}. \quad (124)$$

The Wiener-Hopf factorization of $f^{(\frac{1}{2}, \frac{1}{2})}(z)$ is

$$f^{(\frac{1}{2}, \frac{1}{2})}(z) = f_+^{(\frac{1}{2}, \frac{1}{2})}(z) \frac{h + h_0}{2h} f_-^{(\frac{1}{2}, \frac{1}{2})}(z), \quad (125)$$

where

$$f_+^{(\frac{1}{2}, \frac{1}{2})}(z) = \frac{1}{\sqrt{1 - zh_0}} \frac{1}{1 - z/h}, \quad f_-^{(\frac{1}{2}, \frac{1}{2})}(z) = \frac{\sqrt{z}}{\sqrt{z - h_0}}. \quad (126)$$

The Fourier coefficients of $\ln f^{(\frac{1}{2}, \frac{1}{2})}$ are

$$(\ln f^{(\frac{1}{2}, \frac{1}{2})})_k = \begin{cases} (h_0^k + 2h^{-k})/(2k) & \text{if } k > 0, \\ \ln\left(\frac{h+h_0}{2h}\right) & \text{if } k = 0, \\ -h_0^{-k}/(2k) & \text{if } k < 0. \end{cases} \quad (127)$$

while the inverse correlation length is

$$\xi^{-1} = (\ln f^{(\frac{1}{2}, \frac{1}{2})})_0 = \ln\left(\frac{h + h_0}{2h}\right). \quad (128)$$

We note that (128) coincides with the result obtained by formally interchanging h_0 with h for quenches from the disordered to the ordered phases (116). The Szëgo part of the constant is given by

$$\exp\left[\sum_{k \geq 1} k (\ln f^{(\frac{1}{2}, \frac{1}{2})})_k (\ln f^{(\frac{1}{2}, \frac{1}{2})})_{-k}\right] = \frac{\sqrt{h}}{\sqrt[4]{1 - h_0^2} \sqrt{h - h_0}}. \quad (129)$$

Substituting (126) and (129) into (113) and (126) we obtain the following result for the asymptotic behaviour of the two-point function

$$\rho^{xx}(\ell, t = \infty) = \sqrt{\frac{h\sqrt{1-h_0^2}}{h+h_0}} e^{-\ell/\xi} + O(e^{-\bar{\kappa}\ell}), \quad (130)$$

where $\bar{\kappa} > \xi^{-1}$. In Fig. 5 the analytic result (130) is compared to a numerical computation of $\rho^{xx}(\ell, t = \infty)$ and the agreement is seen to be excellent.

4.4. A general formula for the correlation length.

The results (78), (95), (116) and (128) can be combined into the following general expression for the inverse correlation length ξ^{-1}

$$\xi^{-1} = - \int_0^{2\pi} \frac{dk}{2\pi} \ln |\cos \Delta_k| + \theta_H(h-1)\theta_H(h_0-1) \ln \min(h_0, h_1), \quad (131)$$

where h_1 is defined in Eq. (24). After simple algebraic manipulations, this expression can be cast in the form reported in (23).

5. Transverse correlations

We now turn to the correlation functions of transverse spins σ_i^z . These are simple because σ_i^z are *local* in the fermionic representation $\sigma_i^z = ia_i^y a_i^x$. Concomitantly, and in contrast to order parameter correlators, exact analytic results have been known for many decades, both in equilibrium [72, 52] and after a quantum quench [52]. Here we summarize these results and point out some interesting features, which as far as we know have not previously been stressed in the literature.

5.1. The one-point function

We first consider the one-point function, *i.e.* the magnetization in the z direction, which is [52]

$$\langle \sigma_i^z \rangle = \langle ia_i^y(t) a_i^x(t) \rangle = - \int_0^\pi \frac{dk}{\pi} \left(\cos \theta_k \cos \Delta_k + \sin \theta_k \sin \Delta_k \cos(2\varepsilon_h(k)t) \right), \quad (132)$$

where $\varepsilon_h(k)$, θ_k , Δ_k and defined in (3), (10) and (11) respectively. Unlike the order parameter, the transverse magnetization has a finite value in the stationary state

$$\lim_{t \rightarrow \infty} \langle \sigma_i^z \rangle = - \int_0^\pi \frac{dk}{\pi} \cos \theta_k \cos \Delta_k. \quad (133)$$

The result (133) agrees with the GGE prediction. The late-time limit of (132) can be determined by a stationary phase approximation. The leading contributions arise from the saddle points at $k = 0$ and $k = \pi$ and a straightforward calculation gives

$$\langle \sigma_i^z \rangle \Big|_{Jt \gg 1} = - \int_0^\pi \frac{dk}{\pi} \cos \theta_k \cos \Delta_k \quad (134)$$

$$+ \frac{h_0 - h}{4\sqrt{\pi}(2hJt)^{\frac{3}{2}}} \left[\frac{\sin(4J|1-h|t + \pi/4)}{\sqrt{|1-h||1-h_0|}} - \frac{\sin(4J(1+h)t - \pi/4)}{\sqrt{1+h(1+h_0)}} \right] + O((Jt)^{-\frac{5}{2}}). \quad (135)$$

5.2. The two-point function

The connected transverse two-point correlator is expressed in terms of the Majorana fermions as (see Eq. (53))

$$\begin{aligned}\rho_c^{zz}(\ell, t) &= \langle \sigma_l^z \sigma_{l+\ell}^z \rangle - \langle \sigma_l^z \rangle^2 \\ &= \langle a_1^y a_{\ell+1}^y \rangle \langle a_1^x a_{\ell+1}^x \rangle - \langle a_1^y a_{\ell+1}^x \rangle \langle a_1^x a_{\ell+1}^y \rangle = f_{-\ell} f_\ell - g_{\ell+1} g_{1-\ell}.\end{aligned}\quad (136)$$

Using Eq. (55), $\rho_c^{zz}(\ell, t)$ can be written as a double integral in momentum space

$$\begin{aligned}\rho_c^{zz}(\ell, t) &= \int_{-\pi}^{\pi} \frac{dk_1 dk_2}{(2\pi)^2} e^{i\ell(k_1 - k_2)} \left\{ \prod_{j=1}^2 \sin \Delta_{k_j} \sin(2\varepsilon_h(k_j)t) \right. \\ &\quad \left. - \prod_{j=1}^2 e^{i\theta_{k_j}} \left[\cos \Delta_{k_j} - i \sin \Delta_{k_j} \cos(2\varepsilon_h(k_j)t) \right] \right\}.\end{aligned}\quad (137)$$

where θ_k , Δ_k and $\varepsilon_h(k)$ are given by (10), (11) and (3) respectively.

5.2.1. The infinite time limit and the GGE. In the limit $t \rightarrow \infty$ at fixed, finite ℓ all oscillating terms in this integral average to zero and we are left with

$$\rho_c^{zz}(\ell, \infty) = - \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{i\ell k} e^{i\theta_k} \cos \Delta_k \int_{-\pi}^{\pi} \frac{dp}{2\pi} e^{i\ell p} e^{-i\theta_p} \cos \Delta_p. \quad (138)$$

This resembles the result for the finite temperature connected 2-point function obtained in [52]

$$\langle\langle \sigma_l^z \sigma_{l+\ell}^z \rangle\rangle - \langle\langle \sigma_l^z \rangle\rangle^2 = - \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{i\ell k} e^{i\theta_k} \tanh\left(\frac{\beta\varepsilon_h(k)}{2}\right) \int_{-\pi}^{\pi} \frac{dp}{2\pi} e^{i\ell p} e^{-i\theta_p} \tanh\left(\frac{\beta\varepsilon_h(p)}{2}\right). \quad (139)$$

Here $\langle\langle \mathcal{O} \rangle\rangle$ denotes the thermal equilibrium expectation value at temperature $1/\beta$. Replacing β with a mode dependent inverse temperature as in (65),

$$\beta\varepsilon_h(k) \longrightarrow \beta_k\varepsilon_h(k) = 2 \operatorname{arctanh}(\cos(\Delta_k)), \quad (140)$$

reduces (139) to (138). This again confirms the notion that the GGE can be thought of in terms of a mode-dependent temperature. In order to determine the large- ℓ behaviour of $\rho_c^{zz}(\ell, \infty)$ we change the integration variable in (138) to $z = e^{ik}$. This results in a contour integral along the unit circle C

$$\begin{aligned}\rho_c^{zz}(\ell, \infty) &= - \frac{(h + h_0)^2}{4hh_0} I_+ I_- , \\ I_{\pm} &= \oint_C \frac{dz}{2\pi i} \frac{z^{\ell-1 \pm 1/2}}{\sqrt{(z - 1/h_0)(h_0 - z)}} \frac{(z - 1/h_1)(h_1 - z)}{\pm(z - h^{\mp 1})},\end{aligned}\quad (141)$$

where h_1 is given in (24). Inside the unit circle there is a branch cut connecting 0 with $\min\{h_0, 1/h_0\}$. In addition there is a pole at $\min\{h, 1/h\}$ in I_+ if $h > 1$ and in I_- if $h < 1$. The leading contributions to the integral arise from the pole and the vicinity of the branch point at $\min\{h_0, 1/h_0\}$. Taking these into account (and assuming that $h_0 h \neq 1$) we obtain

$$\rho_c^{zz}(\ell, \infty) = \begin{cases} C_1^{zz} \frac{e^{-2|\ln h_0|\ell}}{\ell} (1 + O(\ell^{-1})) & \text{if } |\ln h| > |\ln h_0|, \\ C_2^{zz} \frac{e^{-(|\ln h| + |\ln h_0|)\ell}}{\sqrt{\ell}} (1 + O(\ell^{-1})) & \text{if } |\ln h_0| > |\ln h|. \end{cases} \quad (142)$$

Here the amplitudes are

$$\begin{aligned} \mathcal{C}_1^{zz} &= \frac{|h_0 - 1/h_0|}{4\pi} \frac{h - h_0}{1 - hh_0} , \\ \mathcal{C}_2^{zz} &= \frac{(h - 1/h)\sqrt{|h_0 - 1/h_0|}(h_0 - h)}{8\sqrt{\pi}h} \sqrt{\frac{h_0 - h}{h_0(hh_0 - 1)}} \frac{e^{\text{sgn}(\ln h)|\ln h_0|/2}}{\sinh \frac{|\ln h| + |\ln h_0|}{2}} . \end{aligned} \quad (143)$$

For quenches such that $hh_0 = 1$ the result is instead

$$\rho_c^{zz}(\ell, \infty) = -\frac{(h - 1/h)^2}{2\pi} e^{-2|\ln h|\ell} + \dots \quad h_0 = 1/h . \quad (144)$$

5.2.2. Time dependence in the space-time scaling regime. We now consider the late time behaviour in the space-time scaling regime, i.e. in an asymptotic expansion around the limit $\ell, t \rightarrow \infty$ at fixed finite ratio $v = \ell/2t$. As $\ell \propto t$ appears in (137) linearly in the phases (albeit in different combinations), the asymptotic power law behaviour can be determined by a stationary phase approximation. The calculation is relatively simple, but the resulting formulas are long and not very illuminating. The general form of the answer is

$$\rho_c^{zz}(\ell, t) \simeq \frac{1}{t} \sum_i \alpha_i(v) \cos(\omega_i(v)t + \varphi_i(v)) , \quad (145)$$

where the sum runs over the various saddle points. The leading power-law t^{-1} comes simply from the stationary phase, but the calculations of the functions α_i , φ_i and ω_i are quite tedious. For this reason we focus on the non-oscillating contribution with $\omega_0(v) = 0$, which we denote by $\rho_0^{zz}(\ell, t)$. This zero-frequency piece arises from terms, in which the phase in Eq. (137) vanishes at the stationary points, i.e. from

$$\int_{-\pi}^{\pi} \frac{dk_1 dk_2}{8\pi^2} e^{2i v t (k_1 - k_2)} \sin \Delta_{k_1} \sin \Delta_{k_2} \cos(2[\varepsilon_h(k_1) - \varepsilon_h(k_2)]t) (1 + e^{i(\theta_{k_1} + \theta_{k_2})}) . \quad (146)$$

The large- t behaviour of this double integral can be extracted by a two-dimensional stationary phase approximation. Its non-oscillating part (characterized by $k_1 = k_2$) for $h \neq 1$ is

$$\rho_0^{zz}(\ell, t) \simeq \frac{1}{4t} \int_0^\pi \frac{dk}{\pi} \sin^2 \Delta_k \cos^2 \theta_k \delta(\varepsilon'_h(k) - v) \quad (147)$$

$$= \frac{\theta_H(v_{\max} - v)}{4\pi t} \left[\frac{\sin^2 \Delta_{k_v} \cos^2 \theta_{k_v}}{|\varepsilon''_h(k_v)|} + \frac{\sin^2 \Delta_{\tilde{k}_v} \cos^2 \theta_{\tilde{k}_v}}{|\varepsilon''_h(\tilde{k}_v)|} \right] , \quad (148)$$

where k_v and \tilde{k}_v are the solutions of $\varepsilon'_h(k) = v$. In the case $h = 1$ there is only a single term in (148) with $k_v = 2\arccos(v/2J)$. The following remarks are in order

- (i) For $v > v_{\max}$ the connected two-point function is exponentially small in $vt = \ell \gg 1$, in accordance with expectations based on causality [11].
- (ii) In contrast to the order parameter two-point function $\rho_c^{xx}(\ell, t)$ studied in paper I, for given t and ℓ only elementary excitations with velocity exactly equal to $v = \pm \ell/2t$ contribute to the asymptotic behaviour of $\rho_c^{zz}(\ell, t)$. This is a consequence of the transverse spins being fermion bilinears.
- (iii) In the limit $v \rightarrow 0$ (147) becomes $(\sin \Delta_k \propto v$, while $\cos^2 \theta_k$ and $\varepsilon''_h(k)$ are finite at the saddle points),

$$\rho_0^{zz}(\ell, t) \propto \ell^2/t^3 . \quad (149)$$

6. Conclusions

In this work we have derived analytic expressions for the stationary behaviour of the reduced density matrix as well as one and two point functions in the transverse field Ising chain after a sudden quench of the magnetic field. We have shown that local observables in the infinite time limit are described by an appropriately defined *generalized Gibbs ensemble*. We have furthermore analyzed the approach in time to the stationary state and addressed the question, how long it takes for the GGE to reveal itself for a given observable. We found that in general this time grows exponentially with the spatial extent of the observable considered, e.g. $t \propto e^{\alpha \ell}$ for a two-point function with spatial separation ℓ . Our analysis can be straightforwardly generalized to quantum quenches in other models with free fermionic spectrum such as the spin-1/2 XY chain in a magnetic field and we expect qualitatively similar behaviour. In contrast, the stationary behaviour in interacting integrable models such as the δ -function Bose gas is not amenable to treatment by the methods employed here and one needs to proceed along different paths [42].

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Appendix A. Asymptotic behaviour of determinants of Toeplitz matrix: Szeg   Lemma, Fisher Hartwig conjecture and generalization

In this appendix we summarize results on Toeplitz determinants, that are used in the main part of the paper. Let $(T_\ell)_{ln} = t_{l-n}$ be a $\ell \times \ell$ Toeplitz matrix with symbol $t(e^{ik})$, *i.e.*

$$t_n \equiv \int_0^{2\pi} \frac{dk}{2\pi} t(e^{ik}) e^{-ink} dk. \quad (\text{A.1})$$

The behaviour of the determinant of T for asymptotically large ℓ depends on the analytic properties of the symbol $t(e^{ik})$.

Appendix A.1. Smooth symbol: The strong Szeg   Lemma

If the symbol $t(z)$ is a *nonzero* continuous function of z on the integration contour, with winding number zero about the origin, the function $\log t(z)$ admits the Fourier (Laurent) expansion

$$\log t(z) = \sum_{q=-\infty}^{\infty} (\log t)_q z^q, \quad (\log t)_q = \int_0^{2\pi} \frac{dk}{2\pi} \log t(e^{ik}) e^{-ikq}. \quad (\text{A.2})$$

The asymptotic behaviour of the determinant is then given by the strong Szeg   limit theorem [68], which states that

$$\det [T_\ell] = E_{[t]} e^{\ell(\log t)_0} (1 + \mathcal{O}(\ell^{1-2\beta})) , \quad (\text{A.3})$$

where

$$E_{[t]} = \exp \left[\sum_{q \geq 1} q (\log t)_q (\log t)_{-q} \right]. \quad (\text{A.4})$$

A convenient alternative form of this result is

$$\ln \det [T_\ell] = \ell \int_0^{2\pi} \frac{dk}{2\pi} \ln t(e^{ik}) + \sum_{q \geq 1} q (\log t)_q (\log t)_{-q} + O(\ell^{1-2\beta}). \quad (\text{A.5})$$

The exponent β characterizing the subleading corrections is determined by the analytic properties of $t(e^{ik})$. The integer part of β is equal to the number of continuous derivatives of $t(e^{ik})$. If the symbol is infinitely differentiable the corrections fall off faster than any power of ℓ .

Appendix A.2. Symbols with zeroes and singularities but zero winding number: The Fisher Hartwig conjecture

One of the many generalizations of the Szëgo theorem is the Fisher Hartwig conjecture. It applies to cases in which the symbol $t(e^{ik})$ has zeroes and/or discontinuities and can be expressed in the form

$$t(e^{ik}) = t_0(e^{ik}) \prod_{r=0}^R e^{i\beta_r(k-k_r-\pi \text{sgn}(k-k_r))} (2 - 2 \cos(k - k_r))^{\alpha_r}. \quad (\text{A.6})$$

Here R is an integer, α_r , β_r , and $0 = k_0 < k_1 < \dots < k_R < 2\pi$ are constants characterizing the location and nature of singularities and zeroes, and $t_0(e^{ik})$ is a smooth nonvanishing function with winding number zero. In terms of the variable $z = e^{ik}$ the symbol is

$$t(z) = t_0(z) z^{\sum_{j=0}^R \beta_j} \prod_{j=0}^R \left(2 - \frac{z_j}{z} - \frac{z}{z_j} \right)^{\alpha_j} g_{\beta_j}(z) z_j^{-\beta_j}, \quad (\text{A.7})$$

where $z_j = e^{ik_j}$, $j = 0, \dots, R$ and

$$g_{\beta_j}(z) = \begin{cases} e^{i\pi\beta_j} & 0 \leq \arg z < k_j \\ e^{-i\pi\beta_j} & k_j \leq \arg z < 2\pi. \end{cases} \quad (\text{A.8})$$

We follow conventions in which $z_0 = 1$ is always included in the set $\{z_j\}$, while for all other z_j we must have either $\alpha_j \neq 0$, $\beta_j \neq 0$ or both. Following the notations of Ref. [69] we introduce the Wiener-Hopf factorization of $t_0(z)$ as

$$t_0(z) = b_+(z) e^{(\log t_0)_0} b_-(z), \\ b_+(z) = e^{\sum_{q \geq 1} (\log t)_q z^q}, \quad b_-(z) = e^{\sum_{q \leq -1} (\log t)_q z^q}. \quad (\text{A.9})$$

Here the constant piece $e^{(\log t_0)_0}$ has been fixed through the requirement $(\ln b_+)_0 = 0$. When $\text{Re}[\alpha_j] > -1/2$ and for $\alpha_j \pm \beta_j \notin \mathbb{Z}^-$, the Fisher-Hartwig conjecture (which under the above conditions has been proved [70]) gives the asymptotic behaviour of the determinant

$$\det [T_\ell] \sim E_{[t_0]} e^{\ell(\log t_0)_0} \left(\prod_{j=0}^R [b_+(z_j)]^{-\alpha_j + \beta_j} [b_-(z_j)]^{-\alpha_j - \beta_j} \right) \ell^{\sum_{j=0}^R (\alpha_j^2 - \beta_j^2)} \\ \times \left[\prod_{0 \leq j < k \leq R} |z_j - z_k|^{2(\beta_j \beta_k - \alpha_j \alpha_k)} \left(\frac{z_k}{z_j e^{i\pi}} \right)^{\alpha_j \beta_k - \alpha_k \beta_j} \right]$$

$$\times \left[\prod_{j=0}^R \frac{G(1 + \alpha_j + \beta_j)G(1 + \alpha_j - \beta_j)}{G(1 + 2\alpha_j)} \right]. \quad (\text{A.10})$$

Here the functional $E_{[t]}$ is given in (A.3) and $G(x)$ is the Barnes G function. In the special case $R = 0$, $\alpha_0 = \beta_0 = 0$ the strong Szëgo lemma applies and the above formula reduces to (A.3). In cases where α_j and β_j don't satisfy the above requirements, the large- ℓ asymptotic behaviour can be determined from the so-called generalized Fisher-Hartwig conjecture. Now there are in general several different *representations* of the symbol in the form (A.6), i.e. there are several possible choices of the parameters $\{\beta_r\}$ in (A.6). The asymptotic behaviour of the determinant is then given as a sum over all inequivalent representations

$$\det [T_\ell] \simeq \sum_{\{\beta_r\}_{\text{ineq}}} \det [T_\ell]_{\{\beta_r\}}, \quad (\text{A.11})$$

where $\det [T_\ell]_{\{\beta_r\}}$ denotes the expression (A.10) for a given set $\{\beta_r\}$. In the cases encountered in the main part of our paper we identify the representations giving rise to the leading asymptotics and quote only their contribution.

Appendix A.3. The case of a symbol with non-zero winding number

A generalization of the Szëgo theorem for symbols with nonzero winding number does exist [71]. If the symbol has negative winding number, i.e. $\exists \kappa \in \mathbb{N}^+$ such that

$$a(e^{ik}) \equiv (-1)^\kappa e^{i\kappa k} t(e^{ik}) \quad (\text{A.12})$$

has winding number zero about the origin, the large- ℓ asymptotics of the determinant is given by [68, 69, 71]

$$\det [T_\ell] = E_{[a]} \exp\left(\ell \int_0^{2\pi} \frac{dk}{2\pi} \ln[a(e^{ik})]\right) \left(\det [\tilde{T}_\kappa] + \mathcal{O}(\ell^{-3\beta})\right) (1 + \mathcal{O}(\ell^{1-2\beta})), \quad (\text{A.13})$$

where $E_{[a]}$ is given by (A.3) and \tilde{T}_κ is a $\kappa \times \kappa$ Toeplitz matrix with elements

$$\left(\tilde{T}_\kappa\right)_{ln} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-i(l-n)k} e^{-i\ell k} \frac{a_-(e^{ik})}{a_+(e^{ik})}, \quad (\text{A.14})$$

where

$$a(e^{ik}) = a_+(e^{ik}) e^{(\log a)_0} a_-(e^{ik}) \quad a_\pm(e^{ik}) = \exp\left[\sum_{j=1}^{\infty} (\ln a)_{\pm j} e^{\pm ijk}\right]. \quad (\text{A.15})$$

As before we use notations where $(\ln a)_j$ are the Fourier coefficients of $\ln a$. For our purposes we only need to consider the case $\kappa = 1$, in which (A.13) takes the simpler form

$$\det T_\ell \sim E_{[a]} e^{\ell(\log a)_0} \int_0^{2\pi} \frac{dk}{2\pi} e^{-i\ell k} \frac{a_-(e^{ik})}{a_+(e^{ik})} + \dots \quad (\text{A.16})$$

At large ℓ we have $\det [T_\ell] \propto e^{\ell/\xi}$, where

$$\xi^{-1} = \lim_{\ell \rightarrow \infty} \frac{\det T_\ell}{\ell} = (\log a)_0 + \lim_{\ell \rightarrow \infty} \frac{1}{\ell} \ln \int_0^{2\pi} \frac{dk}{2\pi} e^{-i\ell k} \frac{a_-(e^{ik})}{a_+(e^{ik})}. \quad (\text{A.17})$$

The case of a symbol with positive winding number can be obtained directly from (A.13) and (A.16) by noting that the transpose of a Toeplitz matrix is another Toeplitz matrix with a symbol of opposite winding number.

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